Chapter 6  Momentum Equation

6.1 Momentum Equation: Derivation
When forces act on a particle, the particle accelerates according to Newton's second law

\[ \sum F = ma \]

\[ \sum F = m \frac{d(v)}{dt} = \frac{d(mv)}{dt} \]

The law can also be formulated for a system composed of a group of particles, for example, a fluid system. In this case, the law may be written as

\[ \sum F = \frac{d(M_{\text{sys}})}{dt} \]

The term \( M_{\text{sys}} \) denotes the total momentum of all mass comprising the system. The above equation is a Lagrangian equation. To derive an Eulerian equation, the Reynolds transport theorem

\[ \frac{dB_{\text{sys}}}{dt} = \frac{d}{dt} \int_{CV} b \rho dA + \int_{CS} b \rho \cdot V \cdot dA \]

Where \( V \) is fluid velocity relative to the control surface at the location where the flow crosses the surface. The extensive property \( B_{\text{sys}} \) becomes the momentum of the system: \( B = M_{\text{sys}} \). The corresponding intensive property \( b \) becomes the momentum per unit mass within the system. The momentum of any fluid particle of mass \( m \) in the system is \( mv \), and so \( b = (mv) / m = v \).

The velocity \( v \) must be relative to an inertial reference frame, that is, a frame that does not rotate and can either be stationary or moving at a constant velocity. Substituting for \( B_{\text{sys}} \) and \( b \)

\[ \frac{d(M_{\text{sys}})}{dt} = \frac{d}{dt} \int_{CV} v \rho dA + \int_{CS} v \rho \cdot V \cdot dA \]

Combining Eqs. above gives the integral form of the momentum equation:

\[ \sum F = \frac{d}{dt} \int_{CV} v \rho dA + \int_{CS} v \rho \cdot V \cdot dA \]

This equation can be expressed in words as

\[ \{ \text{sum of forces acting on the matter in the control volume} \} = \{ \text{time rate of change of momentum in control volume} \} + \{ \text{net outflow rate of momentum through control surface} \} \]

It is important to remember that the momentum equation is a vector equation; that is, there is a direction associated with each term in the equation.

If the flow crossing the control surface occurs through a series of inlet and outlet ports and if the velocity \( v \) is uniformly distributed across each port, then a simplified form of the Reynolds transport theorem, can be used, and the momentum equation becomes

\[ \sum F = \frac{d}{dt} \int_{CV} v \rho dA + \sum_{CS} \dot{m}_o v_o - \sum_{CS} \dot{m}_i v_i \]

where the subscripts \( o \) and \( i \) refer to the outlet and inlet ports, respectively. This form of the momentum equation will be identified as the vector form. Notice that the product of \( \dot{m}v \)
corresponds to the mass per unit time times velocity, or momentum per unit time, which has the same units as force.

As long as \( v \) is uniformly distributed across control surface, Eq. above applies to any control volume, including one that is moving, deforming, or both. In all cases, is the rate at which mass is passing across the control surface, and \( v \) is velocity evaluated at the control surface with respect to the inertial reference frame that is selected.

\[
\begin{align*}
  x - \text{direction:} & \quad \sum F_x = \frac{d}{dt} \int_{CV} \mathbf{v}_x \rho d\mathbf{a} + \sum_{CS} \dot{m}_{ox} \mathbf{v}_x - \sum_{CS} \dot{m}_i \mathbf{v}_{lx} \\
  y - \text{direction:} & \quad \sum F_y = \frac{d}{dt} \int_{CV} \mathbf{v}_y \rho d\mathbf{a} + \sum_{CS} \dot{m}_{oy} \mathbf{v}_y - \sum_{CS} \dot{m}_i \mathbf{v}_{ly} \\
  z - \text{direction:} & \quad \sum F_z = \frac{d}{dt} \int_{CV} \mathbf{v}_z \rho d\mathbf{a} + \sum_{CS} \dot{m}_{oz} \mathbf{v}_z - \sum_{CS} \dot{m}_i \mathbf{v}_{lz}
\end{align*}
\]

where the subscripts \( x, y, \) and \( z \) refer to the force and velocity components in the coordinate directions. These equations will be identified as the component form of the momentum equation. When velocity \( v \) varies across the control surface, the general form of the momentum equation must always be used.

### 6.2 Momentum Equation: Interpretation

#### Force Terms

Consider flow inside a vertical pipe (Fig a). One possible control volume is a cylinder with diameter \( D \) and length \( L \) located just inside the pipe wall. As shown in (Fig. b), the fluid within the control volume has been isolated from its surroundings, and the effect of the surroundings are shown as forces. The effect of the wall is replaced by a force equal to the shear stress (\( \tau \)) times the pipe surface area (\( A_s = \pi DL \)). The force due to pressure is given by pressure (\( p \)) times the section area (\( A = \pi D^2/4 \)) and always acts toward the control surface (a compressive force). The weight of the fluid is given by \( W = \gamma(\pi D^2/4)L \). Thus, the net force acting in the \( z \)-direction is given by

\[
\sum F_z = (p_1 - p_2) \frac{\pi}{4} D^2 - \tau(\pi DL) - \gamma \frac{\pi}{4} D^2 L
\]

Another possible control volume has a length \( L \) and a diameter that is larger than the pipe's outside diameter. As shown in (Fig. c), this control volume cuts through the pipe wall. Comparing (Figs. b and c) shows that the pressure forces are the same. However, in (Fig. c), there is no force associated with shear stress, but there are two new forces, \( F_1 \) and \( F_2 \), which
represent the forces due to the pipe wall. Also, the weight of matter within the control volume now includes the weight of the fluid and the pipe wall \( (W_p) \). The net z-direction force is

\[
\sum F_z = (p_1 - p_2) \frac{\pi}{4} D^2 - F_1 + F_2 - (W_p + \gamma \frac{\pi}{4} D^2 L)
\]

The choice of control volume depends on what information being sought.
To relate the pressure change between sections to wall shear stress?
To find the tensile force carried by the pipe wall?
The sketches shown in (Figs. b and c) are identified as force diagrams (FD). A force diagram shows the forces acting on the matter contained within a control volume. A force diagram is equivalent to a free-body diagram at the instant in time when the momentum equation is applied. In Fig. b, the force of gravity (weight) acts on each mass element in the control volume (with the resultant force acting at the mass center). A force that acts on mass elements within the body is defined as a body force. A body force can act at a distance without any physical contact. Examples of body forces include gravitational, electrostatic, and magnetic forces. Except for the body force (weight), all forces shown in (Figs. b and c) are surface forces. A surface force is defined as a force that requires physical contact, meaning that surface forces act at the control surface.

**Momentum Accumulation**

The term \( \frac{d}{dt} \int_C \mathbf{v} d\mathcal{V} \) represents the rate at which the momentum of the material inside the control volume is changing with time. In particular, the mass of a volume element in the control volume is \( \rho d\mathcal{V} \), so the product \( \mathbf{v} d\mathcal{V} \) is the momentum of a volume element. Integrating over the control volume gives total momentum of the material in the control volume. Taking the time derivative gives the rate at which the momentum is changing. This term may be described as the net rate of momentum accumulation, and it will be referred to as the momentum accumulation term. The units are momentum per unit time, which are equivalent to the units for force.

In many problems, the momentum accumulation is zero. For example, consider steady flow through the control volume surrounding the nozzle shown in Fig. The fluid inside the control volume has momentum because it is moving. However, the velocity and density at each point do not change with time, so the total momentum in the control volume is constant, and the momentum accumulation term is zero. The evaluation of the momentum accumulation term is completed by considering the structural elements (i.e., the nozzle walls). Since the structural elements are stationary, there is no momentum change, so the momentum accumulation rate is zero.

In summary, the momentum of the material inside a control volume is evaluated by integrating the momentum of each volume element over the control volume. If the momentum in each differential volume is constant with time (e.g., steady flow, a stationary structural part), the momentum accumulation rate is zero.
**Momentum Diagram**
The momentum terms on the right side of momentum Eq. may be visualized with a *momentum diagram* (MD). The momentum diagram is created by sketching a control volume and then drawing a vector to represent the momentum accumulation term and a vector to represent momentum flow at each section where mass crosses the control surface. Although the momentum diagram applies to the integral form of the momentum principle, the diagram takes on a simple form when the velocity \( v \) is uniformly distributed across each inlet and outlet port. For example, consider steady flow through the nozzle shown above. For the control volume indicated, the momentum accumulation term is zero, and this vector is omitted from the diagram. If the velocity is assumed to be uniform across the inlet and exit sections, the outlet momentum flow is given by \( \dot{m}_o v_o \) and the inlet momentum flow is given by \( \dot{m}_i v_i \). To evaluate the momentum flow, one can use the diagram to see that

\[
\sum_{CS} \dot{m}_o v_o = (\dot{m}_o v_o \cos \theta) \mathbf{i} + (\dot{m}_o v_o \sin \theta) \mathbf{j}
\]

And

\[
\sum_{CS} \dot{m}_i v_i = (\dot{m}_i v_i) \mathbf{j}
\]

Recognizing that \( \dot{m}_o = \dot{m}_i = \dot{m} \), the above equations can be combined to show that the net outward flow of momentum is

\[
\sum_{CS} \dot{m}_o v_o - \sum_{CS} \dot{m}_i v_i = (\dot{m} v_o \cos \theta) \mathbf{i} + (\dot{m} v_o \sin \theta - \dot{m} v_i) \mathbf{j}
\]

**Systematic Approach**

**Problem Setup**
- Select an appropriate control volume. Sketch the control volume and coordinate axes. Select an inertial reference frame.
- Identify governing equations. This will include either the vector or component form of the momentum equation. Other equations, such as the Bernoulli equation and/or the continuity equation, may be needed.

**Force Analysis and Diagram**
- Sketch body force(s) (usually only gravitational force) on the force diagram.
- Sketch surface forces on the force diagram; these are forces caused by pressure distribution, shear stress distribution, and supports and structures.

**Momentum Analysis and Diagram**
- Evaluate the momentum accumulation term. If the flow is steady and other materials in the control volume are stationary, the momentum accumulation is zero. Otherwise, the momentum accumulation term is evaluated by integration, and an appropriate vector is added to the momentum diagram.
- Sketch momentum flow vectors on the momentum diagram. For uniform velocity, each vector is \( \dot{m} v \).
Q6.77) A windmill is operating in a 10 m/s wind that has a density of 1.2 kg/m³. The diameter of the windmill is 4 m. The constant-pressure (atmospheric) streamline has a diameter of 3 m upstream of the windmill and 4.5 m downstream. Assume that the velocity distributions are uniform and the air is incompressible. Determine the thrust on the mill.

Continuity principle
\( Q_1 = Q_2 \) since density is constant
\( V_1 A_1 = V_2 A_2 \)
\( V_2 = 10 \times (3/4.5)^2 = 4.44 \) m/s

**Momentum principle (x-direction)**

\[
\sum F_x = \dot{m}(v_2 - v_1)
\]
\( F_x = \dot{m}(v_2 - v_1) = (1.2)(\pi/4 \times 3^3)(10)(4.44 - 10) \)
\( F_x = -472.0 \) N (acting to the left)

\[ T = 472 \) N (acting to the right) \]

Q6.64) This “double” nozzle discharges water (at 10°C) into the atmosphere at a rate of 0.50 m³/s. If the nozzle is lying in a horizontal plane, what x-component of force acting through the flange bolts is required to hold the nozzle in place? Note: Assume irrotational flow, and assume the water speed in each jet to be the same. Jet A is 10 cm in diameter, jet B is 12 cm in diameter, and the pipe is 30 cm in diameter.

**solution**

\( V_A = V_B \) given

\( Q_{\text{total}} = Q_A + Q_B \) for incompressible flow

\( Q_{\text{total}} = V_A A_A + V_B A_B \)

\( V_A = V_B = Q_{\text{total}}/(A_A + A_B) \)

\( = 0.5/(\pi \times 0.05 \times 0.05 + \pi \times 0.06 \times 0.06) = 26.1 \) m/s

\( V_1 = 0.5/(\pi \times 0.15 \times 0.15) = 7.07 \) m/s

Bernoulli equation {between 1 & A [same as 1 & B] } 

\( p_1 = (1000/2)(26.12 - 7.072) = 315, 612 \) Pa
Momentum principle (x-direction)

\[ \sum F_x = \dot{m}_o v_{ox} - m_i v_{ix} \]

\[ F_x + p_1 A_1 \sin 30 = -\dot{m} v_A - \dot{m} v_i \sin 30 \]

\[ F_x = -315,612 \times \pi \times 0.15^2 \times \sin 30^\circ - 26.1 \times 1,000 \times 26.1 \times \pi \times 0.05^2 - 7.07 \times 1000 \times 0.5 \sin 30^\circ = 18,270 \text{ N} = -18.27 \text{ kN} \]

Q 6.86
A cart is moving along a track at a constant velocity of 5 m/s as shown. Water (\( \rho = 1000 \text{ kg/m}^3 \)) issues from a nozzle at 10 m/s and is deflected through 180° by a vane on the cart. The cross-sectional area of the nozzle is 0.0012 m². Calculate the resistive force on the cart.

Velocity analysis

\[ V_1 = v_1 = v_2 = 5 \text{ m/s} \]
\[ \dot{m} = p A_1 v_1 \]
\[ = (1000)(0.0012)(5) \]
\[ = 6 \text{ kg/s} \]

Momentum principle (x-direction)

\[ \sum F_x = \dot{m}(v_2 - v_1) \]
\[ -F_r = 6(-5 - 5) = -60 \text{ N} \]
\[ F_r = 60 \text{ N (acting to the left)} \]
6.30 A vane on this moving cart deflects a 10 cm water ($\rho = 1000 \text{ kg/m}^3$) jet as shown. The initial speed of the water in the jet is 20 m/s, and the cart moves at a speed of 3 m/s. If the vane splits the jet so that half goes one way and half the other, what force is exerted on the vane by the jet?

**Momentum principle (x-direction)**

$$F_x = \dot{m}_2 v_2 - \dot{m}_1 v_1$$

$$F_x = (17^2 \cos 45^\circ)(1000)(17)(0.1^2)/2 - (17)(1000)(17)(0.1^2)/2$$

$$= +802 - 270 = -1470 \text{ N}$$

**Momentum principle (y-direction)**

$$F_y = \dot{m}_2 v_2 - \dot{m}_3 v_3$$

$$= (17)(1,000)\sin 45^\circ)(17)(0.1^2)/2 - (17^2)(1000)(0.1^2)/2$$

$$= -333 \text{ N}$$

$$\boxed{F_{\text{on vane}} = (1470\hat{i} + 333\hat{j}) \text{ N}}$$

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**Equation of motion of a rocket**

$$\sum F = p_d A_d - W - D$$

$$m_r \frac{dv_r}{dt} = T - D - W$$

$T$: thrust of the rocket, the sum of the momentum outflow and the pressure force at the nozzle exit. Neglecting the drag and weight, the equation of motion reduces to

$$T = m_r \frac{dv_r}{dt}$$

$$\dot{m}_r = m_i - \dot{m} \ t$$

where $m_i$ is the initial rocket mass and $t$ is the time from ignition.
\[ v_{bo} = \frac{T}{\dot{m}} \frac{1}{\ln \left( \frac{M_i}{M_f} \right)} \]

Where \( v_{bo} \) is the burnout velocity and \( mf \) is the final (or payload) mass. The ratio \( T/\dot{m} \) is known as the specific impulse, \( I_{sp} \), and has units of velocity.

**Q6.89** It is common practice in rocket trajectory analyses to neglect the body-force term and drag, so the velocity at burnout is given by \( v_{bo} = \frac{T}{\dot{m}} \ln \left( \frac{M_i}{M_f} \right) \)

Assuming a thrust-to-mass-flow ratio of 3000 N.s/kg and a final mass of 50 kg, calculate the initial mass needed to establish the rocket in an earth orbit at a velocity of 7200 m/s.

\[ 7200 = 3000 \ln \left( \frac{M_i}{50} \right) \]
\[ \ln \left( \frac{M_i}{50} \right) = \frac{7200}{3000} = 72/30 = 2.4 \]
\[ \frac{M_i}{50} = \exp (2.4) \]
\[ M_i = 50 \exp (2.4) = 550.2 \text{ kg} \]

**Water Hammer: Physical Description**

Whenever a valve is closed in a pipe, a positive pressure wave is created upstream of the valve and travels up the pipe at the speed of sound. If the pressure is greater than the existing steady-state pressure. This pressure wave may be great enough to cause pipe failure. Therefore, a basic understanding of this process, which is called *water hammer*, is necessary for the proper design and operation of such systems.

Consider flow in the pipe shown in Fig. 6.7. Initially the valve at the end of the pipe is only partially open (Fig. 6.7a); consequently, an initial velocity \( V \) and initial pressure \( p_0 \) exist in the pipe. At time \( t = 0 \) it is assumed that the valve is instantaneously closed, thus creating a pressure increase behind the valve and a pressure wave that travels from the valve toward the reservoir at the speed of sound, \( c \). All the water between the pressure wave and the upper end of the pipe will have the initial velocity \( V \), but all the water on the other side of the pressure wave (between the wave and the valve) will be at rest. This condition is shown in Fig. 6.7b. Once the pressure wave reaches the upper end of the pipe (after time \( t = L/c \)), it can be visualized that all of the water in the pipe will be under a pressure \( p_0 + \Delta p \); however, the pressure in the reservoir at the end of the pipe is only \( p_0 \). This imbalance of pressure at the reservoir end causes the water to flow from the pipe back into the reservoir with a velocity \( V \). Thus a new pressure wave is formed that travels toward the valve end of the pipe (Fig. 6.7c), and the pressure on the reservoir side of the wave is reduced to \( p_0 \). When this wave finally reaches the valve, all the water in the pipe is flowing toward the reservoir with a velocity \( V \). This condition is only momentary, however, because the closed valve prevents any sustained flow.
Water hammer process.
(a) Initial condition. (b) Condition during time $0 < t < L/c$. (c) Condition during time $L/c < t < 2L/c$. (d) Condition during time $2L/c < t < 3L/c$. (e) Condition during time $3L/c < t < 4L/c$. 

Diagram showing the water hammer process with different conditions at various time intervals.
Next, during time $2L/c < t < 3L/c$, a rarefied wave of pressure ($p < p_0$) travels up to the reservoir, as shown in Fig. 6.7d. When the wave reaches the reservoir, all the water in the pipe has a pressure less than that in the reservoir. This imbalance of pressure causes flow to be established again in the entire pipe, as shown in Fig. 6.7f, and the condition is exactly the same as in the initial condition (Fig. 6.7a). Hence the process will repeat itself in a periodic manner.

From this description, it may be seen that the pressure in the pipe immediately upstream of the valve will be alternately high and low, as shown in Fig. 6.7a. A similar observation for the pressure at the midpoint of the pipe reveals a more complex variation of pressure with time, as shown in Fig. 6.8b. Obviously, a valve cannot be closed instantaneously, and viscous effects, which were neglected here, will have a damping effect on the process. Therefore, a more realistic pressure–time trace for the point just upstream of the valve is given in Fig. 6.8c. The finite time of closure erases the sharp discontinuities in the pressure trace that were present in Fig. 6.8a. However, it should be noted that the maximum pressure developed at the valve will be virtually the same as for instantaneous closure if the time of closure is less than $2L/c$. That is, the change in pressure will be the same for a given change in velocity unless the negative wave from the reservoir mitigates the positive pressure, and it takes a time $2L/c$ before this negative wave can reach the valve. The value $2L/c$ is called the critical time of closure and is given the symbol $t_c$. 